

# Multipartite Generating Functions and Infinite Products for Quantum Invariants

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## Abstract

We show that multipartite generation functions can be written in terms of the Bell polynomials (known as Faà di Bruno's formula) and the Ruelle spectral functions, whose spectrum is encoded in the Patterson-Selberg function of the hyperbolic three-geometry. We derive an infinite-product formula for the Chern-Simons partition functions and analyze appropriate q-series which leads to the construction of knot invariants. With the help of the Ruelle spectral functions symmetric and modular properties in infinite-product structure can be described.

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*Dedicated to the memory of our friend and colleague, Petya Kulish*

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## 1 Multipartite generating functions

Let us consider, for any ordered  $m$ -tuple of nonnegative integers not all zeros,  $(k_1, k_2, \dots, k_m) = \vec{k}$  (referred to as " $m$ -partite" or *multipartite* numbers), the (multi)partitions, i.e. distinct representations of  $(k_1, k_2, \dots, k_m)$  as sums of multipartite numbers. Let us call  $\mathcal{C}_-^{(z;m)}(\vec{k}) = \mathcal{C}_-^{(z;m)}(z; k_1, k_2, \dots, k_m)$  the number of such multipartitions, and introduce in addition the symbol  $\mathcal{C}_+^{(z;m)}(\vec{k}) = \mathcal{C}_+^{(z;m)}(z; k_1, k_2, \dots, k_m)$ . Their generating functions are defined by [1]

$$\mathcal{F}(z; X) := \prod_{\substack{k_1 \geq 0, \dots, k_m \geq 0, \\ k_1 + \dots + k_m > 0}} (1 - zx_1^{k_1} x_2^{k_2} \dots x_m^{k_m})^{-1} = \sum_{\vec{k} \geq 0} \mathcal{C}_-^{(z;m)}(\vec{k}) x_1^{k_1} x_2^{k_2} \dots x_m^{k_m}, \quad (1.1)$$

$$\mathcal{G}(z; X) := \prod_{\substack{k_1 \geq 0, \dots, k_m \geq 0, \\ k_1 + \dots + k_m > 0}} (1 + zx_1^{k_1} x_2^{k_2} \dots x_m^{k_m}) = \sum_{\vec{k} \geq 0} \mathcal{C}_+^{(z;m)}(\vec{k}) x_1^{k_1} x_2^{k_2} \dots x_m^{k_m}. \quad (1.2)$$

Therefore,

$$\begin{aligned} \log \mathcal{F}(z; X) &= - \sum_{\substack{k_1 \geq 0, \dots, k_m \geq 0, \\ k_1 + \dots + k_m > 0}} \log(1 - zx_1^{k_1} x_2^{k_2} \dots x_m^{k_m}) = \sum_{\vec{k} \geq 0} \sum_{n=1}^{\infty} \frac{z^n}{n} x_1^{nk_1} x_2^{nk_2} \dots x_m^{nk_m} \\ &= \sum_{n=1}^{\infty} \frac{z^n}{n} (1 - x_1^n)^{-1} (1 - x_2^n)^{-1} \dots (1 - x_m^n)^{-1} \\ &= \sum_{n=1}^{\infty} \frac{z^n}{n} \prod_{j=1}^m (1 - x_j^n)^{-1}. \end{aligned} \quad (1.3)$$

Finally,  $\log \mathcal{G}(-z; X) = \log \mathcal{F}(z; X)$ . Let  $\beta_m(n) := \prod_{j=1}^m (1 - x_j^n)^{-1}$ , then

$$\mathcal{F}(z; X) = \sum_{\substack{k_1 \geq 0, \dots, k_m \geq 0, \\ k_1 + \dots + k_m > 0}} \mathcal{C}_-^{(z;m)}(\vec{k}) x_1^{k_1} x_2^{k_2} \dots x_m^{k_m} = \exp \left( \sum_{n=1}^{\infty} \frac{z^n}{n} \beta_m(n) \right), \quad (1.4)$$

$$\mathcal{G}(z; X) = \sum_{\substack{k_1 \geq 0, \dots, k_m \geq 0, \\ k_1 + \dots + k_m > 0}} \mathcal{C}_+^{(z;m)}(\vec{k}) x_1^{k_1} x_2^{k_2} \dots x_m^{k_m} = \exp \left( \sum_{n=1}^{\infty} \frac{(-z)^n}{n} \beta_m(n) \right). \quad (1.5)$$

It is known that the Bell polynomials are very useful in many problems in combinatorics. We would like to note their application in multipartite partition problem [1]. The Bell polynomials technique can be used for the calculation  $\mathcal{C}_-^{(m)}(\vec{k})$  and  $\mathcal{C}_+^{(m)}(\vec{k})$ . Let

$$\mathcal{F}(z; X) := 1 + \sum_{j=1}^{\infty} \mathcal{P}_j(x_1, x_2, \dots, x_m) z^j, \quad \mathcal{P}_j = 1 + \sum_{\substack{k_1 \geq 0, \dots, k_m \geq 0, \\ k_1 + \dots + k_m > 0}} P(\vec{k}; j) x_1^{k_1} \dots x_m^{k_m}, \quad (1.6)$$

$$\mathcal{G}(z; X) := 1 + \sum_{j=1}^{\infty} \mathcal{Q}_j(x_1, x_2, \dots, x_m) z^j, \quad \mathcal{Q}_j = 1 + \sum_{\substack{k_1 \geq 0, \dots, k_m \geq 0, \\ k_1 + \dots + k_m > 0}} Q(\vec{k}; j) x_1^{k_1} \dots x_m^{k_m}. \quad (1.7)$$

Useful expressions for the recurrence relation of the Bell polynomial  $Y_n(g_1, g_2, \dots, g_n)$  and generating function  $\mathcal{B}(z)$  have the forms [1]:

$$Y_{n+1}(g_1, g_2, \dots, g_{n+1}) = \sum_{k=0}^n \binom{n}{k} Y_{n-k}(g_1, g_2, \dots, g_{n-k}) g_{k+1}, \quad (1.8)$$

$\mathcal{B}(z) = \sum_{n=0}^{\infty} Y_n z^n / n! \implies \log \mathcal{B}(z) = \sum_{n=1}^{\infty} g_n z^n / n!$ . To verify the last formula we need to differentiate with respect to  $z$  and observe that a comparison of the coefficients of  $z^n$  in the resulting equation produces an identity equivalent to (1.8). From Eq. (1.8) one can obtain the following explicit formula for the Bell polynomials (it is known as Faa di Bruno's formula)

$$Y_n(g_1, g_2, \dots, g_n) = \sum_{\mathbf{k} \vdash n} \frac{n!}{k_1! \dots k_n!} \prod_{j=1}^n \left( \frac{g_j}{j!} \right)^{k_j}. \quad (1.9)$$

Setting  $X = (x_1, x_2, \dots, x_m, 0, 0, \dots)$ ; for finite additive manner, then the following result holds (see for detail [2]):

$$\mathcal{P}_j = \frac{1}{j!} Y_j(0! \beta_m(1), 1! \beta_m(2), \dots, (j-1)! \beta_m(j)), \quad (1.10)$$

$$\mathcal{Q}_j = \frac{1}{(-1)^j j!} Y_j(-0! \beta_m(1), -1! \beta_m(2), \dots, -(j-1)! \beta_m(j)). \quad (1.11)$$

and

$$\begin{aligned} \mathcal{F}(z; X) &= 1 + \sum_{j=1}^{\infty} \mathcal{P}_j(x_1, x_2, \dots, x_m) z^j \\ &= 1 + \sum_{j=1}^{\infty} \frac{z^j}{j!} Y_j(0! \beta_m(1), 1! \beta_m(2), \dots, (j-1)! \beta_m(j)), \end{aligned} \quad (1.12)$$

$$\begin{aligned} \mathcal{G}(z; X) &= 1 + \sum_{j=1}^{\infty} \mathcal{Q}_j(x_1, x_2, \dots, x_m) z^j \\ &= 1 + \sum_{j=1}^{\infty} \frac{(-1)^j z^j}{j!} Y_j(-0! \beta_m(1), -1! \beta_m(2), \dots, -(j-1)! \beta_m(j)). \end{aligned} \quad (1.13)$$

## 1.1 Restricted specializations

For some specializations, when  $X = (x_1, x_2, \dots, x_m, 0, 0, \dots) = (\underbrace{q, q, \dots, q}_m, 0, 0, \dots)$  we get

$$\mathcal{F}(z; X) = \prod_{\substack{k_1 \geq 0, \dots, k_m \geq 0, \\ k_1 + \dots + k_m > 0}} (1 - zq^{k_1 + k_2 + \dots + k_m})^{-1} = \exp \left( - \sum_{n=1}^{\infty} \frac{z^n}{n} (1 - q^n)^{-m} \right), \quad (1.14)$$

$$\mathcal{G}(z; X) = \prod_{\substack{k_1 \geq 0, \dots, k_m \geq 0, \\ k_1 + \dots + k_m > 0}} (1 + zq^{k_1 + k_2 + \dots + k_m}) = \exp \left( - \sum_{n=1}^{\infty} \frac{(-z)^n}{n} (1 - q^n)^{-m} \right). \quad (1.15)$$

**Spectral functions of hyperbolic three-geometry.** Interesting combinatorial identities may be obtained by applying Euler-Poincaré formula to graded algebras, for example, to the subalgebras of Kac-Moody algebras (see, for example, [3]). From the point of view of the applications, homologies associated with algebras  $\mathfrak{g} = \mathfrak{sl}(N; \mathbb{C})$  important since they constitute the thechnical basis of the proof of the combinatorial identities of Euler-Gauss-Jacobi-MacDonald.

Let us begin by explaining the general lore for the  $\mathfrak{g}$ -structure on compact groups. We recall some results on the Ruelle (Patterson-Selberg type) spectral functions. For details we refer the reader to [4, 5] where spectral functions of hyperbolic three-geometry were considered in connection with three-dimensional Euclidean black holes, pure supergravity, and string amplitudes.

Let  $\Gamma^\gamma \in G = SL(2, \mathbb{C})$  be the discrete group defined by

$$\begin{aligned} \Gamma^\gamma &= \{ \text{diag}(e^{2n\pi(\text{Im } \vartheta + i\text{Re } \vartheta)}, e^{-2n\pi(\text{Im } \vartheta + i\text{Re } \vartheta)}) : n \in \mathbb{Z} \} = \{ \gamma^n : n \in \mathbb{Z} \}, \\ \gamma &= \text{diag}(e^{2\pi(\text{Im } \vartheta + i\text{Re } \vartheta)}, e^{-2\pi(\text{Im } \vartheta + i\text{Re } \vartheta)}). \end{aligned} \quad (1.16)$$

One can construct a zeta function of Selberg-type for the group  $\Gamma^\gamma \equiv \Gamma_{(\alpha, \beta)}^\gamma$  generated by a single hyperbolic element of the form  $\gamma_{(\alpha, \beta)} = \text{diag}(e^z, e^{-z})$ , where  $z = \alpha + i\beta$  for  $\alpha, \beta > 0$ . Actually  $\alpha = 2\pi \text{Im } \vartheta$  and  $\beta = 2\pi \text{Re } \vartheta$ . The Patterson-Selberg spectral function  $Z_{\Gamma^\gamma}(s)$  and its logarithm for  $\text{Re } s > 0$  can be attached to  $H^3/\Gamma^\gamma$  as follows:

$$Z_{\Gamma^\gamma}(s) := \prod_{k_1, k_2 \geq 0} [1 - (e^{i\beta})^{k_1} (e^{-i\beta})^{k_2} e^{-(k_1 + k_2 + s)\alpha}], \quad (1.17)$$

$$\log Z_{\Gamma^\gamma}(s) = -\frac{1}{4} \sum_{n=1}^{\infty} \frac{e^{-n\alpha(s-1)}}{n [\sinh^2(\frac{\alpha n}{2}) + \sin^2(\frac{\beta n}{2})]}. \quad (1.18)$$

The zeros of  $Z_{\Gamma^\gamma}(s)$  are precisely the set of complex numbers  $\zeta_{n, k_1, k_2} = -(k_1 + k_2) + i(k_1 - k_2)\beta/\alpha + 2\pi i n/\alpha$ , with  $n \in \mathbb{Z}$ . The magnitude of the zeta-function is bounded for both  $\text{Re } s \geq 0$  and  $\text{Re } s \leq 0$ , and its growth can be estimated as

$$|Z_{\Gamma^\gamma}(s)| \leq \left( \prod_{k_1 + k_2 \leq |s|} e^{|s|^\ell} \right) \left( \prod_{k_1 + k_2 \geq |s|} (1 - e^{(|s| - k_1 - k_2)\ell}) \right) \leq C_1 e^{C_2 |s|^3} \quad (1.19)$$

for suitable constants  $\ell, C_1, C_2$ . The first product on the right-hand side of (1.19) gives the exponential growth, while the second product is bounded. The spectral function  $Z_{\Gamma\gamma}(s)$  is an entire function of order three and of finite type which can be written as a Hadamard product [5]

$$Z_{\Gamma\gamma}(s) = e^{Q(s)} \prod_{\zeta \in \Sigma} \left(1 - \frac{s}{\zeta}\right) \exp\left(\frac{s}{\zeta} + \frac{s^2}{2\zeta^2} + \frac{s^3}{3\zeta^3}\right), \quad (1.20)$$

where  $\Sigma$  is the set of zeroes  $\zeta := \zeta_{n,k_1,k_2}$  and  $Q(s)$  is a polynomial of degree at most three. (The product formula for entire function (1.20) is also known as Weierstrass formula (1876).)

Let us introduce next the Ruelle spectral function  $\mathcal{R}(s)$  associated with hyperbolic three-geometry [4, 5]. The function  $\mathcal{R}(s)$  is an alternating product of more complicate factors, each of which is so-called Patterson-Selberg zeta-functions  $Z_{\Gamma\gamma}$  [6]. Functions  $\mathcal{R}(s)$  can be continued meromorphically to the entire complex plane  $\mathbb{C}$ , poles of  $\mathcal{R}(s)$  correspond to zeros of  $Z_{\Gamma\gamma}(s)$ .

$$\begin{aligned} \prod_{n=\ell}^{\infty} (1 - q^{an+\varepsilon}) &= \prod_{p=0,1} Z_{\Gamma\gamma}(\underbrace{(al+\varepsilon)(1-i\varrho(\vartheta)) + 1 - a}_{s} + a(1+i\varrho(\vartheta)p)^{(-1)^p}) \\ &= \mathcal{R}(s = (al+\varepsilon)(1-i\varrho(\vartheta)) + 1 - a), \end{aligned} \quad (1.21)$$

$$\begin{aligned} \prod_{n=\ell}^{\infty} (1 + q^{an+\varepsilon}) &= \prod_{p=0,1} Z_{\Gamma\gamma}(\underbrace{(al+\varepsilon)(1-i\varrho(\vartheta)) + 1 - a}_{s} + i\sigma(\vartheta) + a(1+i\varrho(\vartheta)p)^{(-1)^p}) \\ &= \mathcal{R}(s = (al+\varepsilon)(1-i\varrho(\vartheta)) + 1 - a + i\sigma(\vartheta)), \end{aligned} \quad (1.22)$$

$$\begin{aligned} \prod_{n=\ell}^{\infty} (1 - q^{an+\varepsilon})^{bn} &= \mathcal{R}(s = (al+\varepsilon)(1-i\varrho(\vartheta)) + 1 - a)^{b\ell} \\ &\times \prod_{n=\ell+1}^{\infty} \mathcal{R}(s = (an+\varepsilon)(1-i\varrho(\vartheta)) + 1 - a)^b, \end{aligned} \quad (1.23)$$

$$\begin{aligned} \prod_{n=\ell}^{\infty} (1 + q^{an+\varepsilon})^{bn} &= \mathcal{R}(s = (al+\varepsilon)(1-i\varrho(\vartheta)) + 1 - a + i\sigma(\vartheta))^{b\ell} \\ &\times \prod_{n=\ell+1}^{\infty} \mathcal{R}(s = (an+\varepsilon)(1-i\varrho(\vartheta)) + 1 - a + i\sigma(\vartheta))^b, \end{aligned} \quad (1.24)$$

being  $q \equiv e^{2\pi i\vartheta}$ ,  $\varrho(\vartheta) = \text{Re } \vartheta / \text{Im } \vartheta$ ,  $\sigma(\vartheta) = (2 \text{Im } \vartheta)^{-1}$ ,  $a$  is a real number,  $\varepsilon, b \in \mathbb{C}$ ,  $\ell \in \mathbb{Z}_+$ .

Obviously,

$$\begin{aligned} \beta_m(n) &= \prod_{j=1}^m (1 - q^{jn})^{-1} \equiv \prod_{j=1}^{\infty} (1 - q^{jn})^{-1} \prod_{j=m+1}^{\infty} (1 - q^{jn}) \\ &= \frac{\mathcal{R}(s = n(m+1)(1-i\varrho(\vartheta)) + 1 - n)}{\mathcal{R}(s = n(1-i\varrho(\vartheta)) + 1 - n)} \end{aligned} \quad (1.25)$$

and

$$\begin{aligned} \mathcal{F}(z; X) &= \prod_{\substack{k_1 \geq 0, \dots, k_m \geq 0, \\ k_1 + \dots + k_m > 0}} (1 - zq^{k_1 + k_2 + \dots + k_m})^{-1} \\ &\stackrel{\text{by (1.14)}}{=} \exp \left( - \sum_{n=1}^{\infty} \frac{z^n \mathcal{R}(s = -in\varrho(\vartheta)(m+1) + nm + 1)}{n \mathcal{R}(s = -in\varrho(\vartheta) + 1)} \right), \end{aligned} \quad (1.26)$$

$$\begin{aligned} \mathcal{G}(z; X) &= \prod_{\substack{k_1 \geq 0, \dots, k_m \geq 0, \\ k_1 + \dots + k_m > 0}} (1 + zq^{k_1 + k_2 + \dots + k_m}) \\ &\stackrel{\text{by (1.15)}}{=} \exp \left( - \sum_{n=1}^{\infty} \frac{(-z)^n \mathcal{R}(s = -in\varrho(\vartheta)(m+1) + nm + 1)}{n \mathcal{R}(s = -in\varrho(\vartheta) + 1)} \right). \end{aligned} \quad (1.27)$$

Also series for  $\mathcal{F}(z; X)$  and  $\mathcal{G}(z; X)$  have the forms (1.14) and (1.15) correspondingly with  $\beta_m(n)$  is given by Eq. (1.25).

**Example:** Let us calculate  $\mathcal{P}_2$  coefficient. With the help of recurrence relation (1.8) we obtain

$$\begin{aligned} 2\mathcal{P}_2 &= (Y_2(\beta_m(1), \beta_m(2)) = Y_2(\beta_m(1)^2 + \beta_m(2)) = \prod_{j=1}^m (1 - q^{j^2})^{-1} + \prod_{j=1}^m (1 - q^{j^2}) \\ &= \frac{\mathcal{R}(s = 2(m+1)(1 - i\varrho(\vartheta)) - 1)^2 + \mathcal{R}(s = 2(1 - i\varrho(\vartheta) - 1))^2}{\mathcal{R}(s = 2(m+1)(1 - i\varrho(\vartheta)) - 1) \cdot \mathcal{R}(s = 2(1 - i\varrho(\vartheta) - 1))}. \end{aligned} \quad (1.28)$$

**Remark 1.1** In the simple case when  $X = (q, 0, 0, \dots)$  and  $z = 1$  we have  $\mathcal{F}(1; q)^{a_k} = \prod_{k \geq 0} (1 - q^k)^{-a_k}$ . There are some expansions which are differ from power series expansions that are useful in imperical studies. Indeed the following result holds (see also [2])

$$\prod_{k=1}^{\infty} (1 - q^k)^{-a_k} = 1 + \sum_{k=1}^{\infty} \mathcal{B}_k q^k, \quad (1.29)$$

$k\mathcal{B}_k = \sum_{j=1}^k \mathcal{D}_j \mathcal{B}_{k-j} q^k$ ,  $\mathcal{D}_j = \sum_{d|j} da_d$ . Here  $a_k$  and  $\mathcal{B}_k$  are integers. Note that if either sequence  $a_k$  or  $\mathcal{B}_k$  is given, the other is uniquely determined by  $c\mathcal{B}_k$  and  $\mathcal{D}_j$ .

## 1.2 The infinite hierarchy

Setting  $zq^{k_1 + \dots + k_m} q^n = z\Omega_{\vec{k}} q^{n_1}$  with  $\Omega_{\vec{k}} = q^{k_1 + \dots + k_m}$  ( $\vec{k} = (k_1, \dots, k_m)$ ) we get

$$G_1(z\Omega_{\vec{k}}; q) := \prod_{n_1=0}^{\infty} (1 - z\Omega_{\vec{k}} q^{n_1}) = (1 - z\Omega_{\vec{k}}) \cdot \mathcal{R}(s = (1 + \overline{\Omega}(z\Omega_{\vec{k}}))(1 - i\varrho\vartheta)), \quad (1.30)$$

where  $\overline{\Omega}(z\Omega_{\vec{k}}) \equiv \log(z\Omega_{\vec{k}})/2i\pi\vartheta$ . Therefore the infinite products can be factorized as

$$\prod_{k_m=0}^{\infty} \prod_{k_{m-1}=0}^{\infty} \dots \prod_{k_1=0}^{\infty} \prod_{n_1=0}^{\infty} (1 - z\Omega_{\vec{k}} q_1^n) = \prod_{\vec{k} \geq \vec{0}} G_1(z\Omega_{\vec{k}}; q). \quad (1.31)$$

We can treat this factorization as a product of  $m$  copies, each of them is  $G_1(z\Omega_{\vec{k}}; q)$  and corresponds to a free two-dimensional conformal field theory. The next step of the iterative loop becomes the Jackson (convergent) double infinite product  $G_2(z; q, p)$  [7]

$$\begin{aligned} G_2(z\Omega_{\vec{k}}; q, p) &= \prod_{n_2, n_1=0}^{\infty} (1 - z\Omega_{\vec{k}} q^{n_1+n_2}) = \prod_{n_2=0}^{\infty} (1 - z\Omega_{\vec{k}} q^{n_2}) \\ &\times \mathcal{R}(s = (1 + \overline{\Omega}(z\Omega_{\vec{k}} q^{n_2})(1 - i\varrho\vartheta))) \end{aligned} \quad (1.32)$$

For the product (1.32) two first order  $q$ - and  $p$ -equations take the forms [8]

$$\begin{aligned} \frac{G_2(\Omega_{\vec{k}}; q, p)}{G_2(q\Omega_{\vec{k}}; q, p)} &= G_1(\Omega_{\vec{k}}; p), & \frac{G_2(\Omega_{\vec{k}}; q, p)}{G_2(p\Omega_{\vec{k}}; q, p)} &= G_1(\Omega_{\vec{k}}; q), \\ \frac{G_2(qp(q\Omega_{\vec{k}})^{-1}; q, p)}{G_2(qp(\Omega_{\vec{k}})^{-1}; q, p)} &= G_2(p(\Omega_{\vec{k}})^{-1}; p), & \frac{G_2(qp(p\Omega_{\vec{k}})^{-1}; q, p)}{G_2(qp(\Omega_{\vec{k}})^{-1}; q, p)} &= G_2(q(\Omega_{\vec{k}})^{-1}; q). \end{aligned} \quad (1.33)$$

Symmetry roperties of Jackson double infinite product  $G_2(z; q, p)$  analogous to (modular) properties of the standard elliptic gamma functions. For  $z \in \mathbb{C}^*$  the order one  $\Gamma_1$  and double (i.e., the order two)  $\Gamma_2$  standard elliptic gamma functions have the forms

$$\begin{aligned} \Gamma_1(z; q, p) &= \prod_{n_1, n_2=0}^{\infty} \left( \frac{1 - z^{-1} q^{n_1+1} p^{n_2+1}}{1 - z q^{n_1} p^{n_2}} \right), \\ \Gamma_2(z; q, p, t) &= \prod_{n_1, n_2, n_3=0}^{\infty} (1 - z^{-1} q^{n_1+1} p^{n_2+1} t^{n_3+1}) (1 - z q^{n_1} p^{n_2} t^{n_3}). \end{aligned} \quad (1.34)$$

The double elliptic gamma function  $\Gamma_2$  has the following interesting modular properties:

$$\begin{aligned} \Gamma_2(z; a, b, c) &= \Gamma_2(z/a; -1/a, b/a, c/a) \cdot \Gamma_2(z/b; a/b, -1/b, c/b) \cdot \Gamma_2(z/c; a/c, b/c, -1/c) \\ &\times \exp\left(\frac{i\pi}{12} B_{44}(z; a, b, c)\right), \end{aligned} \quad (1.35)$$

where  $B_{44}$  is given by

$$B_{44}(z; a, b, c) = \lim_{x \rightarrow 0} \frac{d^4}{dx^4} \left( \frac{x^4 e^{zx}}{(e^{ax} - 1)(e^{bx} - 1)(e^{cx} - 1)} \right) \quad (1.36)$$

and  $2i\pi a = \log q$ ,  $2i\pi b = \log p$ ,  $2i\pi c = \log t$ . In the case when  $q = p = t$  we get

$$\begin{aligned} \Gamma_1(z; q, q) &= \prod_{n_2=0}^{\infty} \prod_{n_1=0}^{\infty} \left( \frac{1 - z^{-1} q^{n_1+1} p^{n_2+1}}{1 - z q^{n_1} p^{n_2}} \right) = \prod_{n_2=0}^{\infty} \left( \frac{1 - z^{-1} q^{n_2+2}}{1 - z q^{n_2}} \right) \\ &\times \left( \frac{\mathcal{R}(s = (n_2 + \overline{\Omega}(z^{-1}; \vartheta) + 2)(1 - i\varrho(\vartheta)))}{\mathcal{R}(s = (n_2 + \overline{\Omega}(z; \vartheta))(1 - i\varrho(\vartheta)))} \right), \end{aligned} \quad (1.37)$$

where  $\overline{\Omega}(z^{\pm 1}; \vartheta) = \pm \log z / 2\pi i \vartheta$ .

$$\begin{aligned}
\Gamma_2(z; q, q, q) &= \prod_{n_2, n_3=0}^{\infty} \prod_{n_1=0}^{\infty} (1 - z^{-1} q^{n_1+n_2+n_3+3}) (1 - z q^{n_1+n_2+n_3}) \\
&= \prod_{n_2, n_3=0}^{\infty} (1 - z^{-1} q^{n_2+n_3+2}) (1 - z q^{n_2+n_3}) \\
&\times \mathcal{R}(s = (n_2 + n_3 + \overline{\Omega}(z^{-1}; \vartheta) + 3)(1 - i\rho(\vartheta))) \\
&\times \mathcal{R}(s = (n_2 + n_3 + \overline{\Omega}(z; \vartheta) + 1)(1 - i\rho(\vartheta))). \tag{1.39}
\end{aligned}$$

## 2 The quantum group invariants

In this Sect. we view correlators in a CS theory as generating series of quantum group invariants weighted by S-functions. The quantum group invariants can be defined over any semi-simple Lie algebra  $\mathfrak{g}$ . In the  $SU(N)$  Chern-Simons gauge theory we study the quantum  $\mathfrak{sl}_N$  invariants, which can be identified as the so-called colored HOMFLY polynomials.

One important corollary of the LMOV conjecture is the possibility to express a Chern-Simons partition function as an infinite product. In this article we derive such a product. During the calculations we use the characters of the symplectic groups. The latter were found by Weyl [9] using a transcendental method (based on integration over the group manifold). However the appropriate characters may also be obtained by algebraic methods [10]. Following [11] we have used algebraic methods. This allows to exploit the Hopf algebra methods to determine (sub)group branching rules and the decomposition of tensor products.

The motivation for studying an infinite-product formula, associated to topological string partition functions, based on a guess on the modular property of partition function, stimulated by properties of S-functions.

**Review of basic tools.** To derive the infinite-product formula, we need some preliminary material. First of all we denote by  $\mathcal{Y}$  the set of all Young diagrams. Let  $\chi_A$  be the character of the irreducible representation of the symmetric group labeled by a partition  $A$ . Given a partition  $\mu$ , define  $m_j = \text{card}(\mu_k = j; k \geq 1)$ . (The order of the conjugate class of type  $\mu$  is given by:  $\mathfrak{z}_\mu = \prod_{j \geq 1} j^{m_j} m_j!$ .) The symmetric power functions of a given set of variables  $X = \{x_j\}_{j \geq 1}$  are defined as the direct limit of the Newton polynomials:  $p_n(X) = \sum_{j \geq 1} x_j^n$ ,  $p_\mu(X) = \prod_{i \geq 1} p_{\mu_i}(X)$ , and we have the following formulae which determine the Schur function and the orthogonality property of the character

$$s_A(X) = \sum_{\mu} \frac{\chi_A(C_\mu)}{\mathfrak{z}_\mu} p_\mu(X), \quad \sum_{\mu} \frac{\chi_A(C_\mu) \chi_B(C_\mu)}{\mathfrak{z}_\mu} = \delta_{A,B}. \tag{2.1}$$

where  $C_\mu$  denotes the conjugate class of the symmetric group  $S_{|\mu|}$  corresponding to partition  $\mu$  (for details see Sect. 3 of [33]).

Given  $X = \{x_i\}_{i \geq 1}$ ,  $Y = \{y_j\}_{j \geq 1}$ , define  $X * Y = \{x_i \cdot y_j\}_{i \geq 1, j \geq 1}$ . We also define  $X^d = \{x_i^d\}_{i \geq 1}$ . The  $d$ -th Adams operation of a Schur function is given by  $s_A(X^d)$ . (An Adams operation



is type of algebraic construction; the basic idea of this operation is to implement some fundamental identities in S-functions. In particular,  $s_A(X^d)$  means operation of a power sum on a polynomial.) We use the following conventions for the notation:

- $\mathcal{L}$  will denote a link and  $L$  the number of components in  $\mathcal{L}$ .
- The irreducible  $U_q(\mathfrak{sl}_N)$  module associated to  $\mathcal{L}$  will be labeled by their highest weights, thus by Young diagrams. We usually denote it by a vector form  $\vec{A} = (A^1, \dots, A^L)$ .
- Let  $\vec{X} = (x_1, \dots, x_L)$  be a set of  $L$  variables, each of which is associated to a component of  $\mathcal{L}$  and  $\vec{\mu} = (\mu^1, \dots, \mu^L) \in \mathcal{Y}^L$  be a tuple of  $L$  partitions. We write:

$$[\vec{\mu}] = \prod_{\alpha=1}^L [\mu^\alpha], \quad \mathfrak{z}_{\vec{\mu}} = \prod_{\alpha=1}^L \mathfrak{z}_{\mu^\alpha}, \quad \chi_{\vec{A}}(C_{\vec{\mu}}) = \prod_{\alpha=1}^L \chi_{A^\alpha}(C_{\mu^\alpha}),$$

$$s_{\vec{A}}(\vec{X}) = \prod_{\alpha=1}^L s_{A^\alpha}(x_\alpha), \quad p_\mu(X) = \prod_{i=1}^{\ell(\mu)} p_{\mu_i}(X), \quad p_{\vec{\mu}}(\vec{X}) = \prod_{\alpha=1}^L p_{\mu^\alpha}(x_\alpha).$$

**The case of links and a knot.** The quantum  $\mathfrak{sl}_N$  invariant for the irreducible module  $V_{A^1}, \dots, V_{A^L}$ , labeled by the corresponding partitions  $A^1, \dots, A^L$ , can be identified as the HOMFLY invariants for the link decorated by  $Q_{A^1}, \dots, Q_{A^L}$ . The quantum  $\mathfrak{sl}_N$  invariants of the link is given by  $P_{\vec{A}}(\mathcal{L}; q, t) = \mathcal{H}(\mathcal{L} \star \otimes_{\alpha=1}^L Q_{A^\alpha})$ . The colored HOMFLY polynomial of the link  $\mathcal{L}$  can be defined by [12]

$$P_{\vec{A}} = q^{-\sum_{\alpha=1}^L k_{A^\alpha} \omega(\mathcal{K}_\alpha)} t^{-\sum_{\alpha=1}^L |A^\alpha| \omega(\mathcal{K}_\alpha)} \langle \mathcal{L} \star \otimes_{\alpha=1}^L Q_{A^\alpha} \rangle, \quad (2.2)$$

where  $\omega(\mathcal{K}_\alpha)$  is the number of the  $\alpha$ -component  $\mathcal{K}_\alpha$  of  $\mathcal{L}$  and the bracket  $\langle \mathcal{L} \star \otimes_{\alpha=1}^L Q_{A^\alpha} \rangle$  denotes the framed HOMFLY polynomial of the satellite link  $\mathcal{L} \star \otimes_{\alpha=1}^L Q_{A^\alpha}$ . We can define the following invariants:

$$W_{\vec{\mu}}(\mathcal{L}; q, t) = \sum_{\vec{A}=(A^1, \dots, A^L)} \left( \prod_{\alpha=1}^L \chi_{A^\alpha}(C_{\mu^\alpha}) \right) P_{\vec{A}}(\mathcal{L}; q, t). \quad (2.3)$$

The Chern-Simons partition function  $W_{CS}^{SL}(\mathcal{L}; q, t)$  and the free energy  $F(\mathcal{L}; q, t)$  of the link  $\mathcal{L}$  are the following generating series of quantum group invariants weighted by Schur functions  $s_{\vec{A}}$  and by the invariants  $W_{\vec{\mu}}$ :

$$W_{CS}^{SL}(\mathcal{L}; q, t) = 1 + \sum_{\vec{A}} P_{\vec{A}}(\mathcal{L}; q, t) s_{\vec{A}}(\vec{X}) = 1 + \sum_{\vec{\mu}} \frac{W_{\vec{\mu}}(\mathcal{L}; q, t)}{\mathfrak{z}_{\vec{\mu}}} p_{\vec{\mu}}(\vec{X}), \quad (2.4)$$

$$F(\mathcal{L}; q, t) = \log W_{CS}(\mathcal{L}; q, t) = \sum_{\vec{\mu}} \frac{F_{\vec{\mu}}(\mathcal{L}; q, t)}{\mathfrak{z}_{\vec{\mu}}} p_{\vec{\mu}}(\vec{X}). \quad (2.5)$$

**From summations to infinite products.** The Chern-Simons theory has been conjectured to be equivalent to a topological string theory  $1/N$  expansion in physics. This duality

conjecture builds a fundamental connection in mathematics. On the one hand, Chern-Simons theory leads to the construction of knot invariants; on the other hand, topological string theory gives rise to Gromov-Witten theory in geometry.

The Chern-Simons/topological string duality conjecture identifies the generating function of Gromov-Witten invariants as Chern-Simons knot invariants [13]. Based on these thoughts, the existence of a sequence of integer invariants is conjectured [13, 14] in a similar spirit to Gopakumar-Vafa setting [15], which provides an essential evidence of the duality between Chern-Simons theory and topological string theory. This integrality conjecture is called the LMOV conjecture. One important corollary of the LMOV conjecture is to express Chern-Simons partition function as an infinite product derived in this article. The motivation of studying such an infinite-product formula is based on a guess on the modularity property of topological string partition function.

Based on LMOV conjecture the infinite product formulae for the case of links,  $W_{CS}^{SL}(\mathcal{L}; q, t; \vec{X})$  and a knot  $W_{CS}^{SL}(\mathcal{K}; q, t; X)$  are given by [16, 17]

$$W_{CS}^{SL}(\mathcal{K}; q, t; X) = \prod_{\mu} \prod_{Q \in \mathbb{Z}/2} \prod_{m=1}^{\infty} \prod_{k=-\infty}^{\infty} \langle 1 - q^{k+mQ} X^{\mu} \rangle^{-m n_{\mu; g, Q}} \quad (2.6)$$

$$W_{CS}^{SL}(\mathcal{L}; q, t; \vec{X}) = \prod_{\vec{\mu}} \prod_{Q \in \mathbb{Z}/2} \prod_{m=1}^{\infty} \prod_{k=-\infty}^{\infty} \langle 1 - q^{k+mQ} \vec{X} \rangle^{-m n_{\vec{\mu}; g, Q}}. \quad (2.7)$$

Here  $\vec{\mu} = (\mu^1, \dots, \mu^L)$ , the length of  $\mu^i$  is  $\ell_i$ ,  $\vec{X} = (x_1, \dots, x_L)$ , and  $n_{\mu; g, Q}$  are invariants related to the integer invariants in the LMOV conjecture. For a given  $\mu$ ,  $n_{\mu; g, Q}$  vanish for sufficiently large  $|Q|$  due to the vanishing property of  $n_{\mu; g, Q}$ . The products involving  $Q$  and  $k$  are finite products for a fixed partition  $\mu$ .

The symmetric product  $\langle 1 - q^{k+mQ} X^{\mu} \rangle$  and the generalized symmetric product  $\langle 1 - q^{k+mQ} \vec{X} \rangle$  in Eqs. (2.6) and (2.7), respectively, are defined by the formulae [17]

$$\langle 1 - \psi X^{\mu} \rangle = \prod_{x_{i_1}, \dots, x_{i_{\ell(\mu)}}} \left( 1 - \psi x_{i_1}^{\mu_1} \cdots x_{i_{\ell(\mu)}}^{\mu_{\ell(\mu)}} \right), \quad (2.8)$$

$$\langle 1 - \psi \vec{X} \rangle = \prod_{\alpha=1}^L \prod_{i_{\alpha,1}, \dots, i_{\alpha, \ell_{\alpha}}} \left( 1 - \psi \prod_{\alpha=1}^L ((x_{\alpha})_{i_{\alpha,1}}^{\mu_1^{\alpha}} \cdots (x_{\alpha})_{i_{\alpha, \ell_{\alpha}}}^{\mu_{\ell_{\alpha}}^{\alpha}}) \right). \quad (2.9)$$

where  $\psi$  is a generic variable. Because of symmetry  $q \rightarrow q^{-1}$ , we have

$$\begin{aligned} \prod_{k=-\infty}^{\infty} (1 - q^{k+mQ} X^{\mu})^{-m n_{\mu; g, Q}} &= (1 - q^{mQ} X^{\mu})^{-m n_{\mu; g, Q}} \cdot \prod_{k=1}^{\infty} (1 - q^{k+mQ} X^{\mu})^{-2m n_{\mu; g, Q}} \\ &= (1 - q^{mQ} X^{\mu})^{-m n_{\mu; g, Q}} \cdot \mathcal{R}(s = (1 + \bar{\Omega}(q^{mQ} \Omega_{X^{\mu}}))(1 - i_{\varrho}(\vartheta)))^{-2m n_{\mu; g, Q}}, \end{aligned} \quad (2.10)$$

where  $\bar{\Omega}(q^{mQ} \Omega_{X^{\mu}}) \equiv \log(q^{mQ} \Omega_{X^{\mu}})/2i\pi\vartheta$ ,  $X^{\mu} \equiv x_{i_1}^{\mu_1} \cdots x_{i_{\ell(\mu)}}^{\mu_{\ell(\mu)}}$ . Therefore,  $W_{CS}^{SL}(\mathcal{K}; q, t; X^{\mu})$

and  $W_{CS}^{SL}(\mathcal{L}; q, t; \vec{X})$  take the form

$$\begin{aligned} W_{CS}^{SL}(\mathcal{K}; q, t; X^\mu) &= \prod_{\mu} \prod_{Q \in \mathbb{Z}/2} \prod_{x_{i_1}, \dots, x_{i_{\ell(\mu)}}} \prod_{m=1}^{\infty} (1 - q^m t^Q X^\mu)^{-mn \vec{\mu}; g, Q} \\ &\times \mathcal{R}(s = (1 + \overline{\Omega}(q^m t^Q \Omega_{X^\mu}))(1 - i\rho(\vartheta)))^{-2mn \vec{\mu}; g, Q} \end{aligned} \quad (2.11)$$

$$\begin{aligned} W_{CS}^{SL}(\mathcal{L}; q, t; X) &= \prod_{\mu} \prod_{Q \in \mathbb{Z}/2} \prod_{\alpha=1}^L \prod_{i_{\alpha,1}, \dots, i_{\alpha, \ell_\alpha}} \prod_{m=1}^{\infty} (1 - q^m t^Q \vec{X})^{-mn \vec{\mu}; g, Q} \\ &\times \mathcal{R}(s = (1 + \overline{\Omega}(q^m t^Q \Omega_{\vec{X}}))(1 - i\rho(\vartheta)))^{-2mn \vec{\mu}; g, Q}. \end{aligned} \quad (2.12)$$

For further simplification we may use Eq. (1.23). Similar calculations in the case of Kauffman polynomials, relative to the orthogonal group, can be found in a recent paper [18].

**Symmetry and modular properties in infinite-product structure.** In this section we discuss a basic symmetric property of infinite-product structure obtained from the LMOV partition function. For this reason we can use functional equations for the spectral Ruelle functions (1.21)–(1.24):

$$\begin{aligned} &\mathcal{R}(s = (z + b)(1 - i\rho(\vartheta)) + i\sigma(\vartheta)) \cdot \mathcal{R}(s = -(1 + z + b)(1 - i\rho(\vartheta)) + i\sigma(\vartheta)) \\ &= q^{-zb - b(b+1)/2} \mathcal{R}(s = -z(1 - i\rho(\vartheta)) + i\sigma(\vartheta)) \cdot \mathcal{R}(s = (1 + z)(1 - i\rho(\vartheta)) + i\sigma(\vartheta)) \\ &= q^{-z(b-1) - b(b+1)/2} \mathcal{R}(s = (1 - z)(1 - i\rho(\vartheta)) + i\sigma(\vartheta)) \cdot \mathcal{R}(s = z(1 - i\rho(\vartheta)) + i\sigma(\vartheta)). \end{aligned} \quad (2.13)$$

The simple case  $b = 0$  in Eq. (2.13) leads to the symmetry  $\vartheta \rightarrow -\vartheta$ , i.e the symmetry  $q \rightarrow q^{-1}$ .

There is also the following symmetry about  $\mu$  and  $Q$   $n_{\mu; g, -Q} = (-1)^{\ell(\mu)} n_{\mu; g, Q}$ , which can be interpreted as the rank-level duality of the  $SU(N)_k$  and  $SU(k)_N$  Chern-Simons gauge theories [17]. Rank-level duality is essentially a symmetry of quantum group invariants relating a labeling color to its transpose [17]. It can be expressed using symmetry about  $\mu$ ,  $Q$ , and modularity properties of Ruelle functions as follows:  $W_{A^t}(s^{-1}, -v) = W_A(s, v)$ , where  $s = q^{1/2}$ ,  $v = t^{1/2}$ . The stronger version is [17, 12, 19]:  $W_{A^t}(s^{-1}, v) = (-1)^{|A|} W_A(s, v)$ ,  $W_A(s, -v) = (-1)^{|A|} W_A(s, v)$ .

## References

- [1] G. E. Andrews, *The Theory of Partitions*, Encyclopedia of Mathematics vol. 2, Addison-Wesley Publishing Company, 1976.
- [2] G. E. Andrews, *q-Series: Their Development and Application in Analysis, Number Theory, Combinatorics, Physics, and Computer Algebra*, Expository Lectures from the CBMS Regional Conference No. 66, Providence, Rh. I.: AMS, 1986.
- [3] D. B. Fuks, *Cohomology of Infinite-Dimensional Lie Algebras*, Contemporary Soviet Mathematics, Consultants Bureau, New York, 1986.

- [4] L. Bonora and A. A. Bytsenko, *Partition Functions for Quantum Gravity, Black Holes, Elliptic Genera and Lie Algebra Homologies*, Nucl. Phys. B **852** (2011) 508-537; [arXiv:hep-th/1105.4571].
- [5] A. A. Bytsenko, M. Chaichian, R. J. Szabo and A. Tureanu, *Quantum Black Holes, Elliptic Genera and Spectral Partition Functions*, IJGMMP **11** (2014) 1450048; [arXiv:hep-th/1308.2177].
- [6] S. J. Patterson and P. A. Perry, *The divisor of the Selberg zeta function for Kleinian groups, with an appendix by Charles Epstein*, Duke Math. J. **106** (2001) 321-390.
- [7] F. H. Jackson, *The basic gamma-function and the elliptic functions*, Proc. Roy. Soc. London A **76** (1905) 127-144.
- [8] V. P. Spiridonov, *Theta hypergeometric integrals*, Algebra i Analiz, **15** (2003) 161215; [arXiv:math.CA/0303205v2].
- [9] H. Weyl, *The Classical Groups Their Invariants and Representations*, 2nd Ed. Princeton, NJ: Princeton University Press, 1946.
- [10] D. E. Littlewood, *Invariant theory, tensors and group characters*, Philosophical Transactions of the Royal Society A **239** (1944) 305-365.
- [11] B. Fauser, P. D. Jarvis and R. C. King, *Plethysms, replicated Schur functions and series, with applications to vertex operators*, J. Phys. A: Math. Theor. **43** (2010) 405202 (30 pp).
- [12] S. Zhu, *Colored HOMFLY Polynomial Via Skein Theory*, JHEP **229** (2013) 1310; [arXiv:math.GT/1206.5886v1].
- [13] H. Ooguri and C. Vafa, *Knot Invariants and Topological Strings*, Nucl. Phys. B. **577** (2000) 419-438; [arXiv:hep-th/9912123].
- [14] J. M. F. Labastida, M. Mariño and C. Vafa. *Knots, links and branes at large N*, JHEP **11** (2000) 007 (46 pages); [arXiv:hep-th/0010102v1].
- [15] R. Gopakumar and C. Vafa, *On the gauge theory/geometry correspondence*, Adv. Theor. Math. Phys. **3** (1999) 1415-1443; [arXiv:hep-th/9811131].
- [16] K. Liu and P. Peng, *Proof of the Labastida-Mariño-Ooguri-Vafa conjecture*, J. Differential Geom. **85** (2010) 479-525; [arXiv:math-ph/0704.1526].
- [17] K. Liu and P. Peng, *New structure of knot invariants*, Communications in Number Theory and Physics **5** (2011) 601615; [arXiv:GT/1012.2636].
- [18] A. A. Bytsenko and M. Chaichian, *S-Functions, Spectral Functions of Hyperbolic Geometry, and Vertex Operators with Applications to Structure for Weyl and Orthogonal Group Invariants*, Nucl. Phys. B **907** (2016) 258-285; [arXiv:hep-th/1602.06704].
- [19] Q. Chen, K. Liu, P. Peng and S. Zhu, *Congruent skein relations for colored HOMFLY-PT invariants and colored Jones polynomials*, arXiv:math.GT/1402.3571v3.